Resplendent models generated by indiscernibles

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JAF 38

May, 30, 2019

Indiscernibles and Recursive Saturation

Definition

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $I \subseteq M$ be linearly ordered by $<$. I is called a *indiscernible sequence*[*n-indiscernible sequence*] if for all $n < \omega$, all increasing n-tuples $\langle a_0, \ldots, a_{n-1} \rangle, \langle b_0, \ldots, b_{n-1} \rangle$ in $[X]^n$, and all formulas [with n-free variables],

$$\mathcal{M} \models \phi(a_0, \ldots, a_{n-1}) \leftrightarrow \phi(b_0, \ldots, b_{n-1}).$$

If T is a theory with built-in Skolem functions and $\mathcal{M} \models T$, then we call the structure $\mathcal{N} \prec \mathcal{M}$ generated by I an *Ehrenfeucht-Mostowski* model.

Question:

- Is there a recursively saturated model of PA generated by indiscernibles?

Theorem (Ramsey's Theorem)

If $k, n < \omega$, then $\aleph_0 \to (\aleph_0)^n_k$.

Theorem (Ehrenfeucht-Mostowski)

Let T be an $\mathcal{L}$-theory with infinite models. For any infinite linear order $(I, <)$, there is $\mathcal{M} \models T$ containing an indiscernible sequence $(c_i : i \in I)$.

Indiscernibles and Recursive Saturation

Definition (expandability)

Let $\mathcal{L}$ be a recursive language and $\mathfrak{A}$ an $\mathcal{L}$-structure. Let R be a new relation symbol.

$\mathfrak{A}$ is *resplendent*(*chronically resplendent*) if
$\bar{a} \in A$ and $\mathrm{Th}(\mathfrak{A}, \bar{a}) + \varphi(\bar{a}, R))$ is consistent $\Rightarrow \exists R^{\mathfrak{A}} \ ((\mathfrak{A}, R^{\mathfrak{A}}) \vDash \varphi(\bar{a}, R^{\mathfrak{A}}))$.

$\mathfrak{A}$ is *chronically resplendent* if
$\bar{a} \in A$ and $\mathrm{Th}(\mathfrak{A}, \bar{a}) + \varphi(\bar{a}, R))$ is consistent $\Rightarrow \exists R^{\mathfrak{A}} \ ((\mathfrak{A}, R^{\mathfrak{A}}) \vDash \varphi(\bar{a}, R^{\mathfrak{A}})$ and $(\mathfrak{A}, R)$ is resplendent).

$\mathfrak{A}$ is *totally resplendent* if $\exists R_0, R_1, R_2, \ldots$ on A such that each expansion $(\mathfrak{A}, R_0, \ldots, R_{n-1})$ is resplendent and if $(\mathfrak{A}, R_0, R_1, \ldots) \vDash \exists R \varphi(\bar{a}, R)$, then there exists $R^{\mathfrak{A}}$ parametrically definable in $(\mathfrak{A}, R_0, R_1, \ldots)$ such that $(\mathfrak{A}, R_0, R_1, \ldots, R^{\mathfrak{A}}) \vDash \varphi(\bar{a}, R^{\mathfrak{A}})$.

Recursive saturation and Resplendency

Fact

For countable recursively saturated structures over a recursive language $\mathcal{L}$,

$$\begin{aligned} \textit{Recursive saturation} &\Rightarrow \textit{Resplendency} \\ &\Rightarrow \textit{Chronic resplendency} \\ &\Rightarrow \textit{Total resplendency} \end{aligned}$$

Theorem

Let $\mathcal{L}$ be a recursive language and $\mathcal{M}$ be a countable recursively saturated $\mathcal{L}$-structure, and $\bar{a} \in M$. Let $\mathcal{L}'$ be a recursive extension of $\mathcal{L} \cup \{\bar{a}\}$ and T a recursively axiomatized $\mathcal{L}'$-theory. Then, if $\mathrm{Th}(\mathcal{M}, \bar{a}) + T$ is consistent, there is an expansion of $(\mathcal{M}, \bar{a})$ to $\mathcal{L}'$ satisfying T that is recursively saturated as an $\mathcal{L}'$-structure.

Recursive saturation and Indiscernibles

Question (D. Marker, S. Smith)

Is there a recursively saturated model of PA *which is generated by a set of indiscernibles?*

Theorem (F. Abramson, J. Knight, 1981)

Every consistent extension of PA *has a countable recursively saturated model generated by a set of indiscernibles. [Knight,Julia,personal letter to A.Macintyre,1981]*

Proof by R. Kossak

Lemma

If $\mathcal{M} \prec \mathcal{N} \models \mathrm{PA}$ *and* $\mathcal{K} = \sup(\mathcal{M})$ *in* $\mathcal{N}$*, then* $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K} \prec_{\mathrm{end}} \mathcal{N}$*.*

Lemma

Let $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ *be nonstandard models of* PA*, and suppose for some* $a \in \mathcal{N}$, $\mathcal{M} = \sup\{(a)_n : n < \omega\}$*, and* $\mathrm{Scl}((a)_i) < (a)_{i+1}$ *for all* $i < \omega$*. If* $\varphi(x, \bar{y})$ *is a formula and* $\bar{b} \in M$ *with* $\bar{b} < (a)_m$ *for some* $m < \omega$*, then* $\mathcal{M} \models \exists x \varphi(x, \bar{b})$ *iff* $\mathcal{N} \models \exists x < (a)_{m+1} \varphi(x, \bar{b})$*, and* $\mathcal{M} \models \forall x \varphi(x, \bar{b})$ *iff* $\mathcal{N} \models \forall x < (a)_{m+1} \varphi(x, \bar{b})$*.*

Lemma

Let $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ *be nonstandard models of* PA*, and suppose for some* $a \in \mathcal{N}$, $\mathcal{M} = \sup\{(a)_n : n < \omega\}$*, and* $\mathrm{Scl}((a)_i) < (a)_{i+1}$ *for all* $i < \omega$*. Then,* $\mathcal{M}$ *is recursively saturated.*

Proof by R. Kossak

Let $\mathcal{L} = \mathcal{L}_{\mathrm{PA}} \cup \{c\}$ and T be the $\mathcal{L}$-theory consisting of PA and the following:

$$\begin{aligned}&\{\varphi((c)_{m_1}, (c)_{m_2}, \ldots, (c)_{m_k}) \leftrightarrow \varphi((c)_{n_1}, (c)_{n_2}, \ldots, (c)_{n_k}) : \\ &\qquad \langle \bar{m} \rangle, \langle \bar{n} \rangle \in [\omega]^{\geq 0}, \varphi \text{ is an } \mathcal{L}_{\mathrm{PA}}\text{-formula}\} \\ &\cup\{(c)_n < (c)_{n+1} : n < \omega\} \\ &\cup\{(c)_n < ((c)_{n+1})_0 : n < \omega\} \\ &\cup\{t(((c)_n)_i) < ((c)_n)_{i+1} : n, i < \omega, t \text{ is an } \mathcal{L}_{\mathrm{PA}}\text{-term}\}\end{aligned}$$

Let $\mathcal{N} \models T$, $\mathcal{M} = \mathrm{Scl}((c)_n : n < \omega)$. Let $\mathcal{M}_n = \sup(\mathrm{Scl}(((c)_n)_i : i < \omega))$ for each $n < \omega$. Then, $\mathcal{M}_n$'s are recursively saturated. And, $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$ is recursively saturated.
Also, $\mathcal{M}$ is generated by the indiscernibles $\langle (c)_n : n < \omega \rangle$.

Schmerl's Answers

Definition

Let Σ be a complete set of $\mathcal{L}$-formulas. Let I be a countable linearly ordered set. If $T = \{\varphi(\bar{i}) : n \in \mathbb{N},\ \varphi(\bar{x}) \in \Sigma, \langle \bar{i} \rangle \in [I]^n\}$ is consistent, then we say Σ is an *indiscernible type*.

Definition

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $I \subseteq A$. If every element of $\mathfrak{A}$ is generated as $t(\bar{i})$ for some β-term t and $\bar{i} \in I^n$ for some $n \in \mathbb{N}$. Then, we say $\mathfrak{A}$ is *β-generated by I*.

Let $\mathcal{L} = (\beta, \ldots)$ be a finite language with a binary function symbol β. Let **CFF** be the set of sentences

$$\forall x_0, \ldots, x_{n-1} \forall y_0, \ldots, y_{n-1} \exists x \left[\bigwedge_{i<j<n} x_i \neq x_j \rightarrow \bigwedge_{i<n} \beta(x_i, x) = y_i \right]$$

Schmerl's Answers

Theorem (J. Schmerl (1985))

Every countable recursively saturated model of **CFF** *is generated by a set of indiscernibles.*

Theorem (J. Schmerl, 1989)

Let $\mathfrak{A}$ be a countable recursively saturated model of **CFF**. *Then there is an indiscernible type Σ in the language $\mathcal{L}$ such that if I is a linearly ordered set with no last element and $\mathfrak{B}$ is generated by I having indiscernible type Σ, then $\mathfrak{B}$ is β-generated by I, totally resplendent, and $\mathfrak{B} \equiv_{\infty,\omega} \mathfrak{A}$ as $\mathcal{L}$-structures.*

Schmerl's Theorem

Theorem (J. Schmerl, 1989)

Let $\mathfrak{A}$ be a countable recursively saturated model of **CFF**. *Then there is an indiscernible type Σ in the language $\mathcal{L}$ such that if I is a linearly ordered set with no last element and $T = \{\varphi(\bar{i}) : n \in \mathbb{N},\ \varphi(\bar{x}) \in \Sigma, \langle \bar{i} \rangle \in [I]^n\}$, then every model $\mathcal{C}$ of T has the elementary substructure $\mathfrak{B}$ which is β-generated by I and totally resplendent, and such that $\mathfrak{B} \equiv_{\infty,\omega} \mathfrak{A}$ as $\mathcal{L}$-structures.*

Proof of Schmerl's theorem

Using the recursive saturation of $\mathfrak{A}$, we assume that

- $\mathfrak{A}$ has a pairing function(a bijection between A^2 and A).
- $\mathfrak{A}$ has a linear order $<$ and satisfies

$$\forall x_0, \ldots, x_{n-1} \forall y_0, \ldots, y_{n-1} \forall z \exists x > z \left[\bigwedge_{i<j<n} x_i \neq x_j \rightarrow \bigwedge_{i<n} \beta(x_i, x) = y_i \right]$$

for $n < \omega$.

- $\mathfrak{A}$ has distinct elements $a_0, a_1, a_2, \cdots$ such that $\beta(a_n, a_0) = a_{n+1}$ for $n > 0$.

Combinatorics

Definition

Let $\mathcal{L}$ be a finite language consisting only of relation symbols among which is the binary relation symbol $<$ for ordering. Let $\mathfrak{A} = (A, <, \ldots)$ is a finite ordered $\mathcal{L}$-structure and f is a function on $[A]^{<\omega}$. We say that f is *homogeneous* on $\mathfrak{A}$ if whenever $X, Y \subseteq A$ and $\mathfrak{A} \upharpoonright X \cong \mathfrak{A} \upharpoonright Y$, then $f(X) = f(Y)$.

Theorem (AH/NR Theorem)

Suppose $\mathfrak{A} = (A, <, ...)$ is a finite ordered $\mathcal{L}$-structure. Then there is a finite ordered $\mathcal{L}$-structure $\mathfrak{B} = (B, <, ...)$ such that whenever $f : [B]^{<\omega} \to \{0, 1\}$, then there is $\mathfrak{A}' \subseteq \mathfrak{B}$ such that $\mathfrak{A}' \cong \mathfrak{A}$ and f is homogeneous on $\mathfrak{A}'$.

Combinatorics

Definition ((G,r)-free)

Let $(A,<)$ be an infinite linearly ordered set. Let $G = \langle g_n : [A]^n \to A \rangle_{1<n<\omega}$ be a sequence of functions. Let Y be a finite subset of A, $a \in A$, $r < \omega$, and $f : [Y]^{\geq r} \to A$, we say f is *coded by a via G* if for all $s \geq r$ and for all $\langle b_0, \ldots, b_{s-1} \rangle \in [Y]^s$,

$$g_{s+1}(b_0, b_1, \ldots, b_{s-1}, a) = f(b_0, b_1, \ldots, b_{s-1}).$$

We also say that a subset $I \subseteq A$ is (G,r)*-free* if it satisfies the following: if Y is a finite subset Y of I and $f : [Y]^{\geq r} \to A$ is a function, then for each $a \in I$ there is $b \in I$ with $b > a$ that codes f via G.

Lemma

Let $r < \omega$, $G = \{g_n : [\omega]^n \to \omega : 1 < n < \omega\}$ and $I \subseteq \omega$ be (G,r)-free. Let $F : [I]^r \to \{0,1\}$. Then there is a (G,r)-free subset J of I such that F is constant on $[J]^r$.

Proof of Schmerl's theorem

Fix some notations:

- For $n \geq 2$, set $\beta(x_0, x_1, \ldots, x_n) = \beta(\beta(x_0, x_1, \ldots, x_{n-1}), x_n)$.
- Define $\beta(a_{n+1}, -) : [A]^{n+1} \to A$ for $0 < n < \omega$.
- $G = \{\beta(a_{n+1}, -) : 0 < n < \omega\}$.
- $d_0, d_1, d_2, \ldots$ is an enumeration of A.
- $\mathcal{L}_0 = \mathcal{L}$ and for each $n < \omega$, $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{I_n, R_n, d_n\}$ where I_n and R_n are new unary relation symbols. $\mathcal{L}_\omega = \bigcup_{n<\omega} \mathcal{L}_n$.
- Let $\langle \varphi_n(x_0, \ldots, x_{n-1}, y) : 0 < n < \omega \rangle$ be a list of $\mathcal{L}_\omega$-formulas such that φ_n is an $(n+1)$-ary $\mathcal{L}_n$-formula and each $\mathcal{L}_\omega$-formula with free variables among $y, x_0, x_1, x_2, \ldots$ is equivalent to one in the list.
- Let $\langle \psi_n(R) : n < \omega \rangle$ be a list of the $(\mathcal{L}_\omega \cup \{R\})$-sentences with R being a new unary relation symbol such that $\psi_n(R)$ is an $\mathcal{L}_n \cup \{R\}$-formula.

Proof of Schmerl's theorem

Construct a sequence of expansions $\mathfrak{A}_n$ of $\mathfrak{A}$, where $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_{n+1} = (\mathfrak{A}_n, I_n, R_n, d_n)$ such that

(1.1) $\mathfrak{A}_{n+1} = (\mathfrak{A}_n, d_n, I_n, R_n)$ is recursively saturated,

(1.2) $I_0 \subseteq A$ and if $n > 0$, $I_n \subseteq I_{n-1}$,

(1.3) for all $\langle b_0, b_1 \rangle \in [I_n]^2, \beta(b_0, b_1) = a_0$ and $\beta(a_0, b_0) = a_1$.

(1.4) I_n is an n-indiscernible sequence in $\mathfrak{A}_n$,

(1.5) If $n > 1$, $\langle b_0, \ldots, b_{n-1} \rangle \in [I_n]^n$, and $\mathfrak{A}_n \models \exists y \varphi_{n-1}(b_0, b_1, \ldots, b_{n-2}, y)$, then

$$\mathfrak{A}_n \models \varphi_{n-1}(b_0, b_1, \ldots, b_{n-2}, \beta(a_n, b_0, b_1, \ldots, b_{n-2}, b_{n-1})), \text{ and}$$

(1.6) If $\mathfrak{A}_n \models \exists R \psi_n(R)$, then $\mathfrak{A}_n \models \psi_n(R_n)$.

(1.7) I_n is (G, n)-free.

Proof of Schmerl's theorem

Suppose we have constructed the sequence $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \ldots$.
Let Σ_ω be the set of all $\mathcal{L}_\omega$-formulas $\varphi(x_0, x_1, \ldots, x_{n-1})$ such that for all sufficiently large r, whenever $\langle b_0, b_1, \ldots, b_{n-1}\rangle \in [I_r]^n$, then $\mathfrak{A}_\omega \models \varphi(\bar{b})$.
Then, Σ_ω is an indiscernible type.
Let I be a linearly ordered set with no last element and let
$T_\omega = \{\varphi(i_0, \ldots, i_{n-1}) : n < \omega, \langle i_0, \cdots, i_{n-1}\rangle \in [I]^n, \varphi(\bar{x}) \in \Sigma_\omega\}$.
Let $\Sigma = \Sigma_\omega \restriction_{\mathcal{L}}$ and $T = \{\varphi(\bar{i}) : n < \omega, \langle \bar{i}\rangle \in [I]^n, \varphi(\bar{x}) \in \Sigma\}$.
Let $\mathcal{C}_\omega$ be a model of T_ω and $\mathfrak{B}_\omega$ be the β-closure of I in $\mathcal{C}_\omega$.
Let $\mathcal{C}$ be a model of T and $\mathfrak{B}$ be the β-closure of I in $\mathcal{C}$.

Theorem

Let $\mathcal{M}$ and $\mathcal{N}$ be recursively saturated structures. If $\mathcal{M} \equiv \mathcal{N}$ and they realize the same types, then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

Proof of Schmerl's theorem

Lemma

$B_\omega(B)$ is an elementary substructure of $\mathcal{C}_\omega(\mathcal{C})$, and so it is a model of $T_\omega(T)$ β-generated by I having indiscernible type $\Sigma_\omega(\Sigma)$.

Lemma

Then, $\mathfrak{A}$ and $\mathfrak{B}$ realize the same $\mathcal{L}$-types.

Lemma

$\mathfrak{B}$ is totally resplendent.

Questions

- Is there a simpler proof of Schmerl's theorem?
- Further characterization of models generated by indiscernibles;
  - If a countable model of **CFF** can be generated by two different sequences of different types, is it recursively saturated?
  - Or, what conditions make the converse of Schmerl' theorem hold ?