ON COUNTABLE CHAINS HAVING DECIDABLE MONADIC THEORY

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Abstract. Rationals and countable ordinals are important examples of structures with decidable monadic second-order theories. A chain is an expansion of a linear order by monadic predicates. We show that if the monadic second-order theory of a countable chain $C$ is decidable then $C$ has a non-trivial expansion with decidable monadic second-order theory.

§1. Introduction. The study of decidability of logical theories is a well-established research topic with numerous applications in Computer Science, in particular in the field of verification. Many techniques have been developed to build larger and larger classes of structures with a decidable theory. For an overview of recent related results in the framework of monadic second order (shortly: MSO) theories we refer e.g., to [4, 24]. It is interesting to explore the limit of specific decidability techniques, and also to prove general results about the frontier between decidability and undecidability.

In particular, Elgot and Rabin ask in [8] whether there exist maximal decidable structures, i.e., structures $M$ with a decidable first-order (shortly: FO) theory and such that the FO theory of any expansion of $M$ by a non-definable predicate is undecidable. This question is still open. Let us mention some partial results: Soprunov proved in [21] that every countable structure in which a regular ordering is interpretable is not maximal. A partial ordering $(B, <)$ is said to be regular if for every $a \in B$ there exist distinct elements $b_1, b_2 \in B$ such that $b_1 < a$, $b_2 < a$, and no element $c \in B$ satisfies both $c < b_1$ and $c < b_2$. As a corollary he also proved that there is no maximal decidable structure if we replace FO by weak MSO logic. In [2] it was shown that there exists a structure $M$ with a decidable MSO theory and such that any expansion of $M$ by a constant symbol has an undecidable FO theory. Paper [1] gives a sufficient condition in terms of the Gaifman graph of $M$ which ensures that $M$ is not maximal. The condition is the following: for every natural number $r$ and every finite set $X$ of elements of the domain $|M|$ of $M$ there exists an element $x \in |M|$ such that the Gaifman distance between $x$ and every element of $X$ is greater than $r$.

In [3] we considered Elgot–Rabin’s question for chains, i.e., linear orderings expanded with monadic predicates, in the framework of MSO theory. The class of chains is interesting with respect to the above results, since on the one hand

Received October 3, 2010.
no regular ordering seems to be interpretable in such structures (this intuition is supported by the fact that the full binary tree is not interpretable in a chain [18]), and on the other hand their associated Gaifman distance is trivial: thus, they do not satisfy the criterion given in [1]. We proved in [3] that for every chain \( M = (A, <, {}\mathcal{P}) \) such that \((A, <)\) contains a sub-interval of type \( \omega \) or \(-\omega\). \( M \) is not maximal with respect to MSO logic, i.e., there exists an expansion \( M' \) of \( M \) by a predicate which is not MSO definable in \( M \) and such that the MSO theory of \( M' \) is recursive in the one of \( M \).

In this paper we prove that this property holds for every infinite countable chain, namely that no infinite countable chain is maximal with respect to MSO logic. The proof relies on the composition method developed by Feferman–Vaught [9], Läuchli [15] and Shelah [20], which reduces the MSO theory of a sum of ordered structures to the one of its components.

The MSO logic of chains has a special interest as it provides prominent examples of decidable MSO theories, and also for the variety of approaches for proving decidability, such as Ehrenfeucht–Fraïssé games, automata, or composition methods (see e.g., [24]). Let us recall some important decidability results. In his seminal paper [5], Büchi proved that languages of \( \omega \)-words recognizable by automata coincide with languages definable in the MSO theory of \((\omega, <)\), from which he deduced decidability of the theory. The result (and the automata method) was then extended to the MSO theory of any countable ordinal [6], to \( \omega_1 \), and to any ordinal less than \( \omega_2 \) [7]. Gurevich, Magidor and Shelah prove [12] that decidability of MSO theory of \( \omega_2 \) is independent of ZFC. Let us mention results for linear orderings beyond ordinals. Using automata, Rabin [17] proved decidability of the MSO theory of the binary tree, from which he deduces decidability of the MSO theory of \( \mathbb{Q} \), which in turn implies decidability of the MSO theory of the class of countable linear orderings. Shelah [20] improved model-theoretical techniques that allow him to reprove almost all known decidability results about MSO theories, as well as new decidability results for the case of linear orderings, and in particular dense orderings. He proved in particular that the MSO theory of \( \mathbb{R} \) is undecidable. The frontier between decidable and undecidable cases was specified in later papers by Gurevich and Shelah [10, 13, 14]; we refer the reader to the survey [11].

§2. Preliminaries. This section contains standard definitions, notations and some useful results.

2.1. Linear orderings and chains. We first recall useful definitions and results about linear orderings. A good reference on the subject is Rosenstein’s book [19].

A linear ordering \( J \) is a total ordering. The order types of \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) are denoted by \( \omega, \zeta \) and \( \eta \), respectively. Given a linear ordering \( J \), we denote by \(-J\) the backwards linear ordering obtained by reversing the ordering relation.

Given two elements \( j, k \) of a linear ordering \( J \), we denote by \([j, k]\) (respectively \((j, k)\)) the interval \([\min(j, k), \max(j, k)]\) (resp. \((\min(j, k), \max(j, k))\)). An ordering is dense if it contains no pair of consecutive elements. An ordering \( I \) is scattered if there is no order-preserving mapping from \( \eta \) into \( I \).

Given an ordering \( Y \) and a sub-ordering \( X \) of \( Y \), we say that \( X \) is dense in \( Y \) if \([x, y] \cap X \neq \emptyset \) for every pair \( x, y \) of distinct elements of \( Y \), and that \( X \) is nowhere dense in \( Y \) if for every open interval \( Z \) of \( Y \), \( X \cap Z \) is not dense in \( Z \).
We say that \( X \) is co-dense in \( Y \) if \( Y \setminus X \) is dense in \( Y \).

In this paper we consider chains (or, labelled linear orderings), i.e., linear orderings \((A, \prec)\) equipped with a function \( f: A \to T \) where \( T \) is a finite (nonempty) set.

Given a dense ordering \( I \), a finite set \( T \), and a coloring \( C: I \to T \), we say that an interval \( J \subseteq I \) is \( C \)-uniform if for every \( t \in T \) the set \( J \cap C^{-1}(t) \) is either empty or dense in \( J \). We shall use the following result (see [15]).

**Proposition 2.1.** Let \( I \) be a dense ordering. For every finite set \( T \) and every coloring \( C: I \to T \), \( I \) contains an infinite \( C \)-uniform interval.

### 2.2. Logic.

Let us briefly recall useful elements of monadic second-order logic, and settle some notations. For more details about MSO logic see e.g., [11, 23]. Monadic second-order logic is an extension of first-order logic that allows to quantify over elements as well as subsets of the domain of the structure. Given a signature \( L \), one can define the set of (MSO) formulas over \( L \) as well-formed formulas that can use first-order variable symbols \( x, y, \ldots \) interpreted as elements of the domain of the structure, monadic second-order variable symbols \( X, Y, \ldots \) interpreted as subsets of the domain, symbols from \( L \), and a new binary predicate \( x \in X \) interpreted as “\( x \) belongs to \( X \)”.

A sentence is a formula without free variable. As usual, we often identify logical symbols with their interpretation. Given a signature \( L \) and an \( L \)-structure \( M \) with domain \( D \), we say that a relation \( R \subseteq D^m \times (2^D)^n \) is (MSO) definable in \( M \) if and only if there exists a formula \( \varphi(x_1, \ldots, x_m, X_1, \ldots, X_n) \) over \( L \) which is true in \( M \) if and only if \((x_1, \ldots, x_m, X_1, \ldots, X_n)\) is interpreted by an \((m + n)\)-tuple of \( R \). Given a structure \( M \) we denote by MSO(\( M \)) (respectively FO(\( M \))) the monadic second-order (respectively first-order) theory of \( M \).

We say that \( M \) is maximal if MSO(\( M \)) is decidable and MSO(\( M' \)) is undecidable for every expansion \( M' \) of \( M \) by a predicate which is not definable in \( M \).

We can identify labelled linear orderings with structures of the form \( M = (A, \prec, P_1, \ldots, P_n) \) where \( \prec \) is a linear relation interpreted as a linear ordering over \( A \), and the \( P_i \)'s denote unary predicates. We use the notation \( \mathcal{P} \) as a shortcut for the \( n \)-tuple \((P_1, \ldots, P_n)\).

Let \( \Sigma \) and \( \Sigma' \) be relational signatures, \( M \) a \( \Sigma \)-structure with domain \( A \) and \( M' \) a \( \Sigma' \)-structure with domain \( A' \). We say that \( M \) is (MSO) interpretable in \( M' \) if there exist a subset \( D \) of \( A' \) and a surjective map \( \mathcal{F}: D \to A \) such that:

- \( D \) is MSO definable in \( M' \);
- The equivalence relation \( EQ_{\mathcal{F}} = \{(x, y) \in A': \mathcal{F}(x) = \mathcal{F}(y)\} \) is MSO definable in \( M' \);
- For every \( m \)-ary symbol \( R \) of \( \Sigma \), there exists a MSO \( \Sigma' \)-formula \( \varphi_R \) such that \( M \models R(\mathcal{F}(a_1), \ldots, \mathcal{F}(a_m)) \iff M' \models \varphi_R(a_1, \ldots, a_m) \) for all \( a_1, \ldots, a_m \in D \).

The following property of interpretations is well-known.

**Lemma 2.2.** If \( M \) is interpretable in \( M' \) then MSO(\( M \)) is recursive in MSO(\( M' \)).

Let us recall the following result.

**Theorem 2.3** (Rabin [17]). MSO(\( \eta, \prec \)) is decidable.

We shall use the following easy corollary of Theorem 2.3.
Corollary 2.4. Let $M = (\eta, <, P_1, \ldots, P_n)$ be such that $(P_1, \ldots, P_n)$ is a partition and every $P_i$ is non-empty and dense in $\eta$. Then MSO($M$) is decidable.

Proof. We prove that MSO($M$) is recursive in MSO($\eta, <$), and use Theorem 2.3. For every $n$, all structures $(\eta, <, P_1, \ldots, P_n)$ such that $(P_1, \ldots, P_n)$ is a partition and every $P_i$ is non-empty and dense in $\eta$, are isomorphic. Moreover there exists an MSO-formula $U(X_1, \ldots, X_n)$ which expresses that $(X_1, \ldots, X_n)$ is a partition and that every $X_i$ is non-empty and dense. Hence for every sentence $\varphi$, we obtain that $M \models \varphi$ iff
\[(\eta, <) \models \exists X_1 \cdots \exists X_n (U(X_1, \ldots, X_n) \land \varphi^*)\]
where $\varphi^*$ is obtained from $\varphi$ by replacing every atomic formula of the form $P_i(x)$ by $x \in X_i$.

2.3. Elements of composition method. In this paper we rely heavily on composition methods, which allow to compute the theory of a sum of structures from the ones of its summands. For an overview of the subject see [4, 22, 16]. In this section we recall useful definitions and results. The quantifier depth of a formula $\varphi$ is denoted by $qd(\varphi)$. Let $n \in \mathbb{N}$, $\Delta$ any finite signature that contains only relational symbols, and $M_1, M_2$ be $\Delta$-structures. We say that $M_1$ and $M_2$ are $n$-equivalent, denoted $M_1 \equiv^n M_2$, if for every sentence $\varphi$ of quantifier depth at most $n$, $M_1 \models \varphi$ iff $M_2 \models \varphi$.

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \mathbb{N}$ and $\Delta$, the set of sentences of quantifier depth $\leq n$ is infinite. However, it contains only finitely many semantically distinct sentences, so there are only finitely many $\equiv^n$-classes of $\Delta$-structures. In fact, we can compute representatives for these classes.

Lemma 2.5 (Hintikka Lemma). For each $n \in \mathbb{N}$ and a finite signature $\Delta$ that contains only relational symbols, we can compute a finite set $H_n(\Delta)$ of $\Delta$-sentences of quantifier depth at most $n$ such that:

- If $\tau_1, \tau_2 \in H_n(\Delta)$ and $\tau_1 \neq \tau_2$, then $\tau_1 \land \tau_2$ is unsatisfiable.
- If $\tau \in H_n(\Delta)$ and $qd(\varphi) \leq n$, then $\tau \rightarrow \varphi$ or $\tau \rightarrow \neg \varphi$. Furthermore, there is an algorithm that, given such $\tau$ and $\varphi$, decides which of these two possibilities holds.
- For every $\Delta$-structure $M$ there is a unique $\tau \in H_n(\Delta)$ such that $M \models \tau$.

Elements of $H_n(\Delta)$ are called $(n, \Delta)$-Hintikka sentences.

Given a $\Delta$-structure $M$ we denote by $T^n(M)$ the unique element of $H_n(\Delta)$ satisfied in $M$ and call it the $n$-type of $M$. Thus, $T^n(M)$ determines (effectively) which sentences of quantifier-depth $\leq n$ are satisfied in $M$.

As a simple consequence, note that the MSO theory of a structure $M$ is decidable if and only if the function $k \mapsto T^k(M)$ is recursive.

The sum of chains corresponds to concatenation. Let us recall a general definition.

Definition 2.6 (sum of chains). Consider an index structure $Ind = (I, <^I)$ where $<^I$ is a linear ordering. Consider a signature $\Delta = \{<, P_1, \ldots, P_l\}$, where $P_i$ are unary predicate names, and a family $(M_i)_{i \in I}$ of $\Delta$-structures $M_i = (A_i; <^i, P_1^i, \ldots, P_l^i)$ with disjoint domains and such that the interpretation $<^i$ of $<$ in each $M_i$ is a linear ordering. We define the ordered sum of the family $(M_i)_{i \in I}$ as the $\Delta$-structure $M = (A; <^M, P_1^M, \ldots, P_l^M)$ where
• $A$ equals the union of the $A_i$’s
• $x <^k y$ holds if and only if $(x \in A_i$ and $y \in A_j$ for some $i <^I j$), or $(x, y \in A_i$ and $x <^I y$)
• for every $x \in A$ and every $k \in \{1, \ldots, l\}$, $P_k(M)(x)$ holds if and only if $M_j \models P_k(x)$ where $j$ is such that $x \in A_j$.

If the domains of the $M_i$ are not disjoint, replace them with isomorphic chains that have disjoint domains, and proceed as before.

We shall use the notation $M = \sum_{i \in I} M_i$ for the ordered sum of the family $(M_i)_{i \in I}$.

If $I = \{1, 2\}$ has two elements, we denote $\sum_{i \in I} M_i$ by $M_1 + M_2$.

We shall use Shelah’s composition method [20, Theorem 2.4] (see also [11, 22]) which allows to reduce the MSO theory of a sum of chains to the MSO theories of the summands and the MSO theory of the index structure.

**Theorem 2.7** (Composition Theorem [20]). There exists a recursive function $f$ and an algorithm which, given $k, l \in \mathbb{N}$, computes the $k$-type of any sum $M = \sum_{i \in I} M_i$ of chains over a signature $\{<, P_1, \ldots, P_l\}$ from the $f(k, l)$-type of the structure

$$(I, <^I, Q_1, \ldots, Q_p)$$

where

$$Q_j = \{i \in I: T^k(M_i) = \tau_j\} \quad j = 1, \ldots, p$$

and $\tau_1, \ldots, \tau_p$ is the list of all $(k, \Delta)$-Hintikka sentences with $\Delta = \{<, P_1, \ldots, P_l\}$.

The two following results ([20, Sections 5 and 6], see also [25, Theorem 5.6 p. 41]) specifies Theorem 2.7 in case $I = \eta$ and all sets $Q_i$ are either empty or dense in $\eta$.

**Theorem 2.8** (Shuffle). Let $k, l \in \mathbb{N}$, and $S$ be a nonempty set of $k$-types over the signature $\Delta = \{<, P_1, \ldots, P_l\}$. For every sum $M = \sum_{i \in I} M_i$, of chains over $\Delta$ such that $S = \{T^k(M_i): i \in \eta\}$, and $\{i \in \eta: T^k(M_i) = \tau\}$ is dense in $\eta$ for every $\tau \in S$, the $k$-type of $M$ is completely determined by $S$, $k$, and $l$. Moreover it can be computed from $S, k$, and $l$. This $k$-type is called shuffle of $S$ and is denoted by $\text{shuffle}(S)$.

### 2.4. Decomposition of a chain.

Let $M$ be a chain and let $\sim$ be an equivalence relation on the domain of $M$. If the $\sim$-equivalence classes are intervals in $M$ we say that $\sim$ is a convex equivalence relation. In this case the set of $\sim$-equivalence classes can be naturally ordered by $i_1 \leq i_2$ iff $\exists x_1 \in i_1 \exists x_2 \in i_2 (x_1 \leq x_2)$. We denote by $M/\sim$ the linear order of $\sim$-equivalence classes. The mapping that assigns to every $x \in M$ its $\sim$-equivalence class is said to be canonical.

Let $\sim$ be a convex equivalence relation on $M$. Then $M = \sum_{i \in M/\sim} M_i$, where $M_i$ is the subchain of $M$ over the equivalence class $i$.

**Lemma 2.9.** If $\sim$ is a convex equivalence relation which is definable in $M$, then

1. $M/\sim$ is interpretable in $M$.
2. Let $\varphi_1, \ldots, \varphi_k$ be sentences in the signature of $M$. Let a chain $C$ be the expansion of $M/\sim$ by unary predicates $Q_{\varphi_1}, \ldots, Q_{\varphi_k}$ defined as

$$Q_{\varphi_i} = \{i \in M/\sim | M_i \models \varphi_{i}\}.$$

Then $C$ is interpretable in $M$. 


§3. Non-maximality for MSO theories of countable chains.

3.1. Main result. An expansion of $M$ by a predicate $R$ is non-trivial if $R$ is not MSO-definable in $M$.

The next theorem is our main result.

**Theorem 3.1.** Let $M = (A, <, \mathcal{F})$ be an infinite countable chain. There exists a non-trivial expansion $M'$ of $M$ by a monadic predicate such that $\text{MSO}(M')$ is recursive in $\text{MSO}(M)$. In particular if $\text{MSO}(M)$ is decidable, then $\text{MSO}(M')$ is decidable.

In this section we prove Theorem 3.1. We shall use the following result from [3].

**Lemma 3.2.** Let $M = (A, <, \mathcal{F})$ be an infinite chain which contains an interval of order type $-\omega$ or $\omega$. There exists a non-trivial expansion $M'$ of $M$ such that $\text{MSO}(M')$ is recursive in $\text{MSO}(M)$.

In the rest of this section we prove Theorem 3.1.

Let $\mathcal{P} = (P_1, \ldots, P_t)$. We can assume w.l.o.g. that $\mathcal{F}$ is a partition of $A$. The structure $M'$ will be defined as the expansion of $M$ with some unary predicate $R$.

Consider the equivalence relation $\approx$ defined on $A$ which holds between $x$ and $y$ if either $[x, y]$ is finite, or $[x, y]$ is contained in an open dense interval which is $C$-uniform with respect to the coloring $C: A \to \{1, \ldots, t\}$ which maps every $x \in A$ to the unique $i$ such that $x \in P_i$. Observe that the relation $x \approx y$ is MSO-definable in $M$. Each $\approx$-equivalence class has one of the following forms:

1. orderings of type $-\omega$, or $\omega$, or $\zeta$;
2. an interval of order type $\eta$ which is $C$-uniform;
3. finite orderings.

We denote by $J$ the linear order $M/\approx$ of the $\approx$-equivalence classes. We can write $M = \sum_{j \in J} M_j$ (respectively $M' = \sum_{j \in J} M'_j$), where for every $j \in J$ the domain of $M_j$ (resp. $M'_j$) corresponds to an $\approx$-equivalence class.

Hence, at least one of the following cases holds:

1. At least one $\approx$-equivalence class has order-type $-\omega$, or $\omega$, or $\zeta$;
2. at least one $\approx$-equivalence class has order-type $\eta$;
3. all $\approx$-equivalence classes are finite.

We prove Theorem 3.1 for each of these cases separately.

If there exists at least one $\approx$-equivalence class of order-type $-\omega$, or $\omega$, or $\zeta$, then the result follows from Lemma 3.2.

The case when at least one $\approx$-equivalence class has order-type $\eta$ is considered in the next subsection. The case when all $\approx$-equivalence classes are finite is considered in subsection 3.3.

3.2. Second case: there exists at least one $\approx$-equivalence class of order type $\eta$.

In this case we can expand $M$ with any predicate $R$ which satisfies the following conditions:

1. If $j$ is an $\approx$-equivalence class of type (1) or (3) then $R \cap j = \emptyset$.
2. If $j$ is an $\approx$-equivalence class of type (2) and $Y_j = \{l \mid P_l \cap j \neq \emptyset\}$, then $R$ is dense and co-dense in $j \cap P_l$ for every $l \in Y_j$.

**Lemma 3.3.** The set $R$ is not definable in $M$. 
Proof. Assume that a unary predicate \( H \) is definable in a chain \( M \) by an MSO formula \( \varphi(x) \). Let \( g \) be an automorphism on \( M \). Then \( H \) should be invariant under \( g \), i.e., \( g \) maps \( H \) onto itself.

Let \( j \) be an \( \approx \)-equivalence class of type (2). It is order-isomorphic to \( \eta \). Let \( M_j \) be the substructure of \( M \) over \( j \). For every \( l \in Y_j \), \( P_l \) is dense and co-dense in \( j \). Then, by (2), there are \( a_1, a_2 \in j \) such that \( a_1 \in P_l \cap R \) and \( a_2 \in P_l \setminus R \) for some \( l \in Y_j \), and there exists an automorphism \( f \) of \( M_j \), which maps \( a_1 \) to \( a_2 \). Hence, \( R \) is not invariant under \( f \). We can extend \( f \) to an automorphism \( g \) of \( M \). Since \( R \) is not invariant under \( g \), we derive that \( R \) is not MSO-definable in \( M \).

The next definition introduces notations which will be used throughout the paper.

Definition 3.4 (Chains \( N_k \) and \( N'_k \)). Let \( M \) be a chain in a signature \( \Delta \) and let \( M' \) be an expansion of \( M \) by a predicate \( R \). For \( k > 0 \) we define chains \( N_k \) and \( N'_k \) as follows. Let \( (J, <) \) be the chain \( M/\approx \) of \( \approx \)-equivalence classes.

1. \( N_k \) is the chain \( (J, <) \) of predicates \( \{ \text{Type}_{k, \tau} \mid \tau \in H_k(\Delta) \} \) defined as: \( \text{Type}_{k, \tau}(j) \) holds iff \( T^k(M_j) = \tau \).
2. \( N'_k \) is the chain \( (J, <) \) of predicates \( \{ \text{Type}'_{k, \tau} \mid \tau \in H_k(\Delta \cup \{ R \}) \} \) defined as: \( \text{Type}'_{k, \tau}(j) \) holds iff \( T^k(M'_j) = \tau \).

Note that \( N_k \) and \( N'_k \) are chains over the same domain, however they have different signature. The following lemma is a consequence of Lemma 2.9.

Lemma 3.5. 1. \( N_k \) is interpretable in \( M \).
2. \( N'_k \) is interpretable in \( M' \).
3. \( N_k \) is interpretable in \( N_m \) for every \( m \geq k \).

Lemma 3.6. MSO(\( M' \)) is recursive in MSO(\( M \)).

Proof. We show how to reduce \( T^n(\( M' \)) \) to MSO(\( M \)) for every \( n \geq 3 \). Note that \( T^0(M') \) is always empty (since there is no sentence with quantifier depth 0 in the signature of \( M' \)), and moreover \( T^1(M') \) and \( T^2(M') \) clearly reduce to \( T^3(M') \). The main reduction steps can be represented as follows:

\[
T^n(M') \rightarrow \text{MSO}(N'_n) \rightarrow \text{MSO}(N_n) \rightarrow \text{MSO}(M).
\]

Let \( n \geq 3 \). By Theorem 2.7, \( T^n(M') \) is recursive in MSO(\( N'_n \)).

By Lemma 3.5 there is an interpretation of \( N_n \) in \( M \). Therefore MSO(\( N_n \)) is recursive in MSO(\( M \)).

It remains to show that MSO(\( N'_n \)) is recursive in MSO(\( N_n \)).

Let us prove that for every \( j \in J \), \( T^n(M'_j) \) can be computed from \( T^n(M_j) \). First of all, using \( T^3(M_j) \) we can check whether the \( \approx \)-class \( j \) has type (2). Indeed, only classes of type (2) are dense, thus

\[
T^3(M_j) \rightarrow \forall x \forall y (x < y \rightarrow \exists z (x < z \land z < y))
\]

iff \( j \) has type (2).

If \( j \) is not of type (2), then by definition of \( R \) we have \( R \cap j = \emptyset \). In this case for every sentence \( \varphi \) we have \( M'_j \models \varphi \) iff \( M_j \models \varphi^* \) where \( \varphi^* \) is obtained from \( \varphi \) by replacing every atomic formula of the form \( R(x) \) by \( \neg(x = x) \). This shows that in this case \( T^n(M'_j) \) can be computed from \( T^n(M_j) \).

Assume now that \( j \) has type (2). Let \( Y_j = \{ i : P_i \cap j \neq \emptyset \} \). The set \( Y_j \) is computable from \( T^1(M_j) \). Let us denote by \( i_1, \ldots, i_t \) the (distinct) elements

- of type (2).
-
of $Y_j$. For $u = 1, \ldots, t$, let $P_{u,1} = P_{u} \cap R$ and $P_{u,2} = P_{u} \setminus R$. It follows from our assumptions that all sets $P_{u,1}$ and $P_{u,2}$ are non-empty and dense in the domain of $M_j$ (which we identify with $\eta$). By Corollary 2.4 it follows that the MSO theory of the structure $S = (\eta, <, P_{u,1}, P_{u,2}, \ldots, P_{1,1}, P_{1,2})$ is decidable. We have $R = \cup \cup_{1 \leq u \leq t} P_{u,1}$, and $P_{u} = P_{u,1} \cup P_{u,2}$, thus $R$ and all predicates $P_{i,u}$ are MSO-definable in $S$. It follows that MSO($M'_n$) is recursive in MSO($S$).

We proved that for every $j \in J$, $T^n(M'_j)$ is computable from $T^n(M_j)$. This implies that every predicate $\text{Type}_{i,k}^j$ in the signature of $N'_n$ is equivalent to a boolean combination of predicates $\text{Type}_{i,k}^{M_j}$ in the signature of $N_n$, and thus is definable in $N_n$. It follows that $N'_n$ is interpretable in $N_n$, and Lemma 2.2 yields that MSO($N'_n$) is recursive in MSO($N_n$). \hfill \triangleright

This completes the proof for the second case.

3.3. Third case: all $\equiv$ equivalence classes are finite. The construction in this case shares some ideas with the previous one but is more involved.

Since every $\equiv$-equivalence class is finite, there are no consecutive $\equiv$-equivalence classes. Therefore, the ordering $J$ of $\equiv$-equivalence classes is infinite, countable and dense (i.e., it is isomorphic with $\eta$, $1 + \eta$, $\eta + 1$ or $1 + \eta + 1$).

We shall expand $M$ with a unary predicate $R$ which will be defined “at the level of $J$”, i.e., for every $j$ we will have $(j \cap R) \in \{\varnothing, j\}$. Thus we actually define a set $R' \subseteq J$, and then define $R$ by: $j \cap R = j$ if $j \in R'$, and $j \cap R = \varnothing$ otherwise.

For every $n > 0$ let $C_n$ denote the coloring which maps every $j \in J$ to $T^n(M_j)$. Consider the equivalence relation $j \sim_n k$ which holds between elements $j, k \in J$ iff $(j = k$, or there exists a $C_n$-uniform open interval of $J$ which contains both $j$ and $k$). Each $\sim_n$-equivalence class is either a singleton, which we call an $n$-point, or a (maximal) $C_n$-uniform open interval, which we call an $n$-interval. If $I$ is an $n$-interval and $S = \{\tau \mid \text{there is } j \in I \text{ such that } T^n(M_j) = \tau\}$, then $I$ is said to be an $S$-interval. Note that if $I$ is an $S$-interval and $\tau \in S$, then $\{j \mid T^n(M_j) = \tau\}$ is dense in $I$.

The main idea is to define $R'$ in such a way that the following property holds:

(*) For every $n > 0$, every $n$-interval $I$ of $J$, and every $n$-type $\tau \in \{T^n(M_j) \mid i \in I\}$, the set $R'$ is both dense and co-dense in $I \cap \{i \mid T^n(M_i) = \tau\}$.

This property will ensure that $R'$ is not definable in $M$ (see Proposition 3.12), and on the other hand, will allow to reduce the computation of the $n$-type of the expansion of $M$ by $R$ to MSO($M$) (see Proposition 3.13).

For every $n > 0$ let $\Pi_n$ (respectively, $I_n$) denote the set of $n$-points (respectively $n$-intervals), and let $\Pi = \bigcup_n \Pi_n$. The definition of $R'$ proceeds in two main stages: we first define the restriction of $R'$ to $J \setminus \Pi$, and then the restriction of $R'$ to $\Pi$ (by defining it on every $\Pi_n$, by induction over $n$).

The following is easy:

**Lemma 3.7** (properties of $\Pi_n$). 1. $\Pi_n$ is MSO definable in $N_n$ for $m \geq n$. 2. $\Pi_n$ is nowhere dense in $J$.

**Proof.** (1) is immediate. (2) Assume for a contradiction that $\Pi_n$ is dense in some open interval $I$ of $J$. Then by Proposition 2.1, the interval $I$ contains some
on countable chains having decidable monadic theory

$C_n$-uniform open subinterval $I'$, and $I'$ is simultaneously contained in some $n$-interval, and contains $n$-points, which is impossible.

First stage: definition of $R'$ on $J \setminus \Pi$:

Let $J' = J \setminus \Pi$. If $J'$ is empty then we are done. Otherwise, let $\approx_1$ be an equivalence relation on $J$ defined as follows: $x \approx_1 y$ if $x = y$ or there is an open interval $I \subseteq J$ of order type $\eta$ such that $x, y \in I$, $M_x$ and $M_y$ are isomorphic, and the set \{ $z \in I \setminus \Pi$ | $M_z$ and $M_y$ are isomorphic $\}$ is dense in $I$. Note that an $\approx_1$-equivalence class is either a singleton or of order type $\eta$.

Now we define $R'$ on $J'$ as any set that contains all singleton $\approx_1$-equivalence classes and is dense and co-dense in every $\approx_1$-equivalence class of order type $\eta$.

The following lemma is crucial in order to prove $(\ast)$.

**Lemma 3.8.** Let $n > 0$. $I$ be an open sub-interval of an $n$-interval, $\tau$ be an $n$-type which appears (densely and co-densely) in $I$, and

$$Y_\tau = I \cap \{ j: T^n(M_j) = \tau \}.$$

If $Y_\tau \cap \Pi = \emptyset$, then every element $x$ of $Y_\tau$ belongs to an $\approx_1$-equivalence class $C$ of order type $\eta$, and $Y_\tau$ has a non-empty intersection with both $R'$ and its complement.

**Proof.** Let $x \in Y_\tau$. The structure $M_x$ is finite. Therefore, there is a sentence $\varphi$ such that a chain satisfying $\varphi$ iff it is isomorphic to $M_x$. Let $m = \text{gd}(\varphi)$. Since $x \notin \Pi_m$, it follows that $x$ belongs to an $m$-interval $I_m$. Hence, the set $E_x = \{ z | M_z$ and $M_x$ are isomorphic $\}$ is dense in $I_m \cap I$. Now every element $z$ of $E_x$ satisfies $T^m(M_z) = x$, thus $E_x \cap I \subseteq Y_\tau$. By our assumption $Y_\tau \cap \Pi = \emptyset$, from which it follows that all elements of $E_x \cap I \cap I_m$ (which is an infinite set) are $\approx_1$-equivalent to $x$. Thus the $\approx_1$-equivalence class of $x$ is of order type $\eta$, hence $E_x \cap I \cap I_m$ has a non-empty intersection with both $R'$ and its complement.

Second stage: definition of $R'$ on $\Pi$:

We define by induction on $n$ the set $R'_n \subseteq \Pi_n \setminus \Pi_{n-1}$, and define then the restriction of $R'$ to $\Pi$ as $\bigcup_{n \geq 0} R'_n$.

Let us first explain the definition informally. We want that eventually $R'$ satisfies $(\ast)$. We start with a simple example. Consider the case $n = 1$, and the partition of $J$ into 1-intervals and 1-points. Consider a 1-interval $I$. It does not contain any 1-point (by definition), but it can contain $m$-points for some $m > 1$. Thus if we want that $(\ast)$ holds for $n = 1$ and $I$, we have to ensure that the definition of $R'$ for $m$-points is compatible with $(\ast)$. If the set $I \cap \Pi$ is finite, or even nowhere dense in $I$, then the definition of $R'$ on $I \cap \Pi$ is then the first stage suffices to ensure that $(\ast)$ holds for $n = 1$. Thus we could simply choose to put all elements of $I \cap \Pi$ in $R'$ (or all in the complement of $R'$). However it can happen that all elements of $I$ belong to $\Pi$. Thus we need some convenient strategy for defining $R'$ on $I \cap \Pi$.

Let us consider the following example.

**Example 3.1.** Let $(A_j)_{j > 0}$ be a family of disjoint subsets of $\eta$. For every $i > 0$ let $A_{i \leq i} = \bigcup_{j=1}^i A_j$. Assume that the two following properties hold:

1. for every $i > 0$, $A_{i \leq i}$ is order-isomorphic to a subset of the integers.
2. $\bigcup_{i > 0} A_i$ is dense in $\eta$. 


We define the sets $R_i \subseteq A_i$ as follows:

- $R_1 = \emptyset$
- for every $i > 0$, $x \in R_{i+1}$ iff $x \in A_{i+1}$ and there are $y, z \in A_{\leq i}$ such that
  1. $y < x < z$ and $(y, z) \cap A_{\leq i} = \emptyset$ (note that this implies that $y$ and $z$ are unique) and
  2. $y$ and $z$ belong to the complement of $\cup_{j=1}^{i} R_j$.

It is easy to see that $R_i$ is MSO-definable in $(\eta, <)$ with parameters $A_1, \ldots, A_i$. Let us show that the set $R = \cup_{i>0} R_i$ is dense and co-dense in the set $A = \cup_{i>0} A_i$.

Take any open interval $I \subseteq \eta$. We have to show that there is a point in $I \cap A \cap R$ and a point in $(I \cap A) \setminus R$.

Toward a contradiction assume that $I \cap A \cap R = \emptyset$. Since $A$ is dense in $\eta$ there is $i$ such that $I \cap A_{<i}$ contains at least two points. These points do not belong to $R$. Let us consider two consecutive points $y < z$ in $I \cap A_{<i}$. Since $I \cap A \cap R = \emptyset$ we obtain that $y, z \notin \cup_{j=1}^{i} R_j$. Since $A$ is dense in $(y, z)$ there is a minimal $m > i$ such that $A_m \cap (y, z) \neq \emptyset$. Then according to the definition of $R_m$ all points of $A_m \cap (y, z)$ should be in $R_m$ and hence in $R$. This contradicts the assumption that $I \cap A \cap R = \emptyset$.

Similar arguments show that $(I \cap A) \setminus R \neq \emptyset$. Hence $R$ is dense and co-dense in $A$.

Our definition of $R_i'$ refines the definition of $R_i$ in the above example. First obstacle we have to overcome is to generalize the definition given for the family $(A_i)_{i>0}$ to the family $(\Pi_i)_{i>0}$. Note that a set $\Pi_i$ is not necessarily order isomorphic to a subset of integers, though $\Pi_i$ is nowhere dense. The second obstacle, in order to prove (*), is that even in the case when $J = \Pi = \eta$, we have to ensure that $R'$ is dense in $\eta \cap Y_x$ and the construction in the example only ensures that $R$ is dense in $\eta$.

For every $n > 1$ let $A_n = \Pi_n \setminus \Pi_{n-1}$. We say that $y$ is an $(m, n)$-left bound for $x$ (and denote it as $BL_m^n(y, x)$) if the following conditions hold:

1. $y < x$
2. $x \in A_n$
3. $y \in \Pi_{n-1}$
4. $(y, x)$ is a subinterval of some $m$-interval and $T^m(M_x) = T^m(M_y)$

Note that by (4) if $BL_m^n(y, x)$ then $m < n$. The predicate $BR_m^n(y, x)$ for the relation "$y$ is an $(m, n)$-right bound for $x$" is defined similarly.

Define $\text{lrank}_m^n(x)$ and $\text{rrank}_m^n(x)$ as

$$\text{lrank}_m^n(x) := \exists y \, BL_m^n(y, x) \land \neg \exists y \, BL_{m+1}^n(y, x).$$

$$\text{rrank}_m^n(x) := \exists y \, BR_m^n(y, x) \land \neg \exists y \, BR_{m+1}^n(y, x).$$

If $\text{lrank}_m^n(x)$ (respectively, $\text{rrank}_m^n(x)$) we say that the left rank (respectively, right rank) of $x$ is $(m, n)$.

We are going to define $R'_n \subseteq A_n$ by induction on $n$.

We say that $R'_m$ holds at the left bound for $x$ if $\text{lrank}_m^n(x)$ and

- either $\{ y \mid BL_m^n(y, x) \}$ has a maximal element $z$ and $z \in \cup_{j=1}^{m-1} R'_j$
- or $\{ y \mid BL_m^n(y, x) \} \cap \cup_{j=1}^{m-1} R'_j$ is co-final in $\{ y \mid BL_m^n(y, x) \}$.

If this happens, we define $R'_m(\eta)$ as the set of all $y \in A_m$ such that $\text{lrank}_m^n(x)$ for some $n > 1$.
One defines similarly “$R'_{<s}$ holds at the right bound for $x$”.
We define the sets $R'_i \subseteq A_i$ as follows:

- $R'_i = \emptyset$.
- For every $i > 0$ and $x \in A_{i+1}$, $x \notin R'_{i+1}$ iff $(\text{lrank}_{m_1}^i(x))$ and $(\text{rrank}_{m_2}^i(x))$ hold for some $m_1, m_2$, and $R'_{<i+1}$ holds at the left bound for $x$, and $R'_{<i+1}$ holds at the right bound for $x$.

Recall that the structure $\mathcal{N}_k$ was defined in Definition 3.4. The following lemma is immediate.

**Lemma 3.9.** For every $m \geq n \geq 1$ the relation $R'_n$ is MSO definable in $\mathcal{N}_m$.

The following lemma describes a property of $R'$ which is an instance of (*) and is central for our proof of (*).

**Lemma 3.10.** Let $n > 0$ and let $I$ be an open subinterval of an $n$-interval. Then $\Pi$ is dense in $\mathcal{I}$. Assume that $\Pi$ is dense in $\mathcal{I}$. Then, $R'$ is both dense and co-dense in $\mathcal{I}$.

**Proof.** We are going to prove that $\mathcal{I}$ contains a point in $R'$ and a point outside $R'$. Since $I$ is an arbitrary non-empty open interval this implies the conclusion of the lemma.

Since $\Pi$ is dense in $\mathcal{I}$, the set $\Pi \cap \mathcal{I}$ contains at least two points $a < b$. Let $a \in A_{k_1} \cap \mathcal{I}$ and $b \in A_{k_2} \cap \mathcal{I}$. Note that $k_1, k_2 > n$.

Toward a contradiction assume

\[(1) \quad R' \cap \mathcal{I} \cap (a, b) = \emptyset.\]

We shall prove the following:

\[(2) \quad \text{for every } s > \max(k_1, k_2) \text{ and every } x \in A_s \cap \mathcal{I} \cap (a, b), \ \text{there exists } l \leq k_1 \text{ such that } \text{lrank}_l^i(x) \text{ holds.}\]

Let $s, x$ be as in (2). Both elements $a$ and $x$ belong to $I$, thus $[a, x]$ is a subinterval of an $n$-interval. Moreover we have $T^n(M_a) = T^n(M_x)$ since $a, x \in \mathcal{I}$. Finally we have $a \in A_{k_1}$ with $k_1 < s$, thus $a \in \Pi_{s-1}$. It follows that $\text{BL}^n_{<a}(a, x)$ holds. Therefore, $(\text{lrank}_l^i(x))$ holds for some $l \geq n$. Since $a \in \Pi_{k_1}$, by condition 4 in the definition of $\text{BL}^i_{<a}$ we obtain that $\neg \text{BL}^n_{<a}(a, x)$. Therefore,

\[(3) \quad \text{If } l > k_1 \text{ and } y \text{ satisfies } \text{BL}^n_{<a}(y, x) \text{ then } y > a.\]

By (1) we have $x \notin R'_s$, therefore by the definition of $R'_s$, we obtain that $R'_{<s}$ holds at the left bound for $x$. If $l > k_1$ then by (3) and (1) it follows that $R'_{<s}$ does not hold at the left bound for $x$. Hence, a contradiction. Therefore, $l \leq k_1$, which yields (2).

Recall that $\Pi$ is dense in $\mathcal{I}$. It follows from (2) that there exists $l_1 \leq k_1$ and a non-empty open interval $V \subseteq (a, b)$ such that $\{x \in \Pi \mid \text{lrank}_l^i(x) \wedge s > k_1\}$ is dense in $V \cap \mathcal{I}$.

Since for every $i$ the set $\Pi_i$ is nowhere dense and $A_{i+1} = \Pi_{i+1} \setminus \Pi_i$, we obtain that for every $r$, there exist integers $s_1 < s_2 < \cdots < s_r$ and elements $x_1 < x_2 < \cdots < x_r$ of $V \cap \mathcal{I}$ such that $\text{lrank}_{l_1}^i(x_i)$ holds for every $i$.

Let $u$ be the number of $(l_1, \{<, \bar{P}\})$-Hintikka sentences, and let $r > 2u$. We obtain that there is an $l_1$-type $\tau'$ and $x_1 < x_2 < \cdots < x_r$ such that

$\tau' = T^{h}(M_{x_1}) = T^{h}(M_{x_2}) = T^{h}(M_{x_r})$. 
First observe
\[(4) \{ y \mid y > x_i \land BL_{1}^{n}(y, x_j) \} \neq \emptyset\]
Indeed, on the one hand if \( BL_{1}^{n}(x_i, x_j) \), then \((x_i, x_j)\) is a sub-interval of an \( l_{1} \)-interval, and therefore \( BL_{l_{1}}^{n}(x_i, x_j) \). Since \( x_i < x_p \) it follows that
\[x_p \in \{ y \mid y > x_i \land BL_{l_{1}}^{n}(y, x_j) \},\]
and therefore \(4\) holds.

On the other hand, if \( \neg BL_{l_{1}}^{n}(x_i, x_j) \) then \((x_i, x_j)\) is not a sub-interval of any \( l_{1} \)-interval, and in this case \( \{ y \mid y > x_i \land BL_{l_{1}}^{n}(y, x_j) \} = \{ y \mid BL_{l_{1}}^{n}(y, x_j) \} \neq \emptyset \).

Next, observe that, by \(1\), no element of \((a, x_j) \cap Y_{1} \) belongs to \( R' \), and hence no element of \((x_i, x_j) \cap Y_{1} \) belongs to \( R' \).

Recall that for every \( s > \max(k_1, k_2) \) and every \( x \in A_{s} \cap Y_{1} \cap (a, b) \) we proved that \( rank_{1}^{l}(x) \) holds for some \( l \geq n \). Therefore, \( l_{1} \in [n, k_{1}] \) and \( \tau' \to \tau \). In addition, \((x_i, x_j)\) is an sub-interval of an \( n \)-interval. therefore \((x_i, x_j) \cap Y_{1} \subseteq (x_i, x_j) \cap Y_{1} \). Hence, no element of \((x_i, x_j) \cap Y_{1} \) belongs to \( R' \), and
\[(5) \{ y \mid y > x_i \land BL_{l_{1}}^{n}(y, x_j) \} \cap R' = \emptyset.\]
Finally by \(4\) and \(5\), \( R'_{x_j} \) does not hold at the left bound for \( x_j \), and by the definition of \( R'_{x_{j}} \), we obtain that \( x_j \in R'_{x_{j}} \). This contradicts \(1\).

We have proved that \( Y_{1} \) contains a point in \( R' \). The proof that \( Y_{1} \) contains a point outside \( R' \) is similar. 

**Lemma 3.11.** \( R' \) satisfies \((*)\).

**Proof.** Let \( n > 0 \) and let \( I \) be some \( n \)-interval. Then \( I \) is an \( S \)-interval for some set \( S = \{ \tau_{1}, \ldots, \tau_{P} \} \) of \( n \)-types. We have to prove that for every \( \tau \in S \) the set \( R' \) is both dense and co-dense in the set \( Y_{1} = I \cap \{ x \mid T^{n}(M, x) = \tau \} \). Let us fix \( \tau \), and let us consider an open interval \( Z \) of \( I \). We shall prove that \( Z \cap Y_{1} \) has a non-empty intersection with both \( R' \) and the complement of \( R' \).

If \( \Pi \) is not dense in \( Z \cap Y_{1} \), then there exists an open interval \( K \) of \( Z \) such that \( K \) contains no element of \( \Pi \). Then, by Lemma 3.8, \( K \cap Y_{1} \) contains both elements from \( R' \) and the complement of \( R' \).

If \( \Pi \) is dense in \( Z \cap Y_{1} \), then the result follows from Lemma 3.10.

**Proposition 3.12.** \( R \) is not MSO definable in \( M \).

**Proof.** The proof is similar to the one of Lemma 3.3. Assume for a contradiction that \( R \) is definable in \( M \) by some formula \( \varphi(x) \) with quantifier depth \( k \). For every \( j \in J \) we have \( j \in R' \) iff there exists \( x \in j \cap R \), i.e., iff \( T^{k+1}(M, x) \to \exists \varphi(x) \). The latter property is expressible in the structure \( N_{k+1} \). Therefore \( R' \) is definable in the structure \( N_{k+1} \) (even with a quantifier-free formula).

Let \( n = k + 1 \), and let \( K \subseteq J \) be an infinite \( C_{n} \)-uniform interval. The set \( R' \) is dense and co-dense in every set \( S_{c} = \{ a \in K \mid C_{n}(a) = c \} \) for \( c \) in the range of \( C_{n} \), thus there exist \( a_{1} \in R' \cap S_{c} \), and \( a_{2} \in S_{c} \cap R' \). Now \( K \) is \( C_{n} \)-uniform, thus there exists an automorphism \( g \) of the sub-structure of \( N_{n} \) with domain \( K \) which maps \( a_{1} \) to \( a_{2} \), and \( R' \) is not invariant under \( g \), which contradicts the fact that \( R' \) is definable in \( N_{n} \).

**Proposition 3.13.** \( \text{MSO}(M') \) is recursive in \( \text{MSO}(M) \).
Proof. We show how to reduce $T^n(M')$ to $\text{MSO}(M)$ for every $n \geq 1$.

Let us denote by $(J_n, <)$ the linear order of $\sim_n$-equivalence classes. Let $\pi_n : J \rightarrow J_n$ denote the corresponding canonical mapping. For $I \in J_n$ let $\delta_n(I) = \cup_{j \in I} j$. Note that $\delta_n(I)$ is an interval in $M$. We denote by $O'_{n, I}$ the subchain of $M'$ over $\delta_n(I)$. Observe that

$$O'_{n, I} = \sum_{j \in I} M'_j$$

and

$$M' = \sum_{i \in J_n} O'_{n, I}$$

Let $O'_n$ be the expansion of $(J_n, <)$ by monadic predicates

$$\{ Q_{n, \tau'} \mid \tau' \in H_n(\Delta \cup \{R\}) \},$$

where $\Delta$ is the signature of $M$, and $Q_{n, \tau'}(I)$ holds iff $T^n(O'_{n, I}) = \tau'$.

The main reduction steps can be represented as follows:

$$T^n(M') \rightarrow \text{MSO}(O'_n) \rightarrow \text{MSO}(N_n) \rightarrow \text{MSO}(M).$$

The first reduction step is a consequence of Theorem 2.7, which shows that the computation of $T^n(M')$ reduces to the one of $T'(n, l)(O'_{n, I})$.

By Lemma 3.5 there is an interpretation of $N_n$ in $M$, therefore $\text{MSO}(N_n)$ is recursive in $\text{MSO}(M)$.

To complete the proof it is sufficient to show that $\text{MSO}(O'_n)$ is recursive in $\text{MSO}(N_n)$. This immediately follows from the next Lemma.

**Lemma 3.14.** There exists an interpretation of $O'_n$ in $N_n$. Moreover, there is an algorithm which computes such an interpretation from $n$.

Proof. We consider the interpretation map $\mathcal{F} = \pi_n$. The domain $D = J$, the relation $EQ_{\mathcal{F}}$, as well as the ordering relation between $\sim_n$-equivalence classes, are definable in $N_n$. Thus it remains to provide an interpretation in $N_n$ of each predicate $Q_{n, \tau'}$, i.e., to show that for every $n$-type $\tau'$ in the signature of $M'$, one can define in $N_n$ the predicate $P_{n, \tau'}$ which holds at $j$ iff $(j \in I \in J_n$ and $T^n(O'_{n, I}) = \tau'$).

First note that for every $\tau \in H_n(\Delta)$ there are $\tau^R, \tau^R \in H_n(\Delta \cup \{R\})$ such that

$$\tau^R \leftrightarrow (\tau \land \forall tR(t)) \text{ and } \tau^R \leftrightarrow (\tau \land \forall t \neg R(t)) \text{.}$$

Moreover, $\tau^R, \tau^R$ are computable from $\tau$.

We claim

$$P_{n, \tau'}(j) \text{ iff } \begin{cases} \tau' = \tau^R \text{ and } j \text{ is an } n\text{-point such that } \\
Type_{n, \tau}(j) \text{ and } j \in R'_{\leq n} \end{cases}$$

$$\begin{cases} \tau' = \tau^R \text{ and } j \text{ is an } n\text{-point such that } \\
Type_{n, \tau}(j) \text{ and } j \notin R'_{\leq n} \end{cases}$$

$$\begin{cases} \tau' = \text{shuffle}(\{ \tau^R \mid \tau \in S \} \cup \{ \tau^R \mid \tau \in S \}). \\
\text{where } S \subseteq H_n(\Delta) \text{ and } j \text{ belongs to an } S\text{-interval} \end{cases}$$

Observe that

- Predicates $Type_{n, \tau}$ for $\tau \in H_n(\Delta)$ are in the signature of $N_n$.
- the set $R'_{\leq n} = \bigcup_{\leq n} R'_i$ is definable in $N_n$, by Lemma 3.9.
• for every subset $S = \{\tau_1, \ldots, \tau_p\}$ of $n$-types in the signature of $M$, the predicate “$j$ belongs to an $S$-interval of $N_n$” is definable in $N_n$.

These observations together with (6) imply that the predicates $P_n, j'$ are definable in $N_n$, thus $O'_n$ is interpretable in $N_n$.

It remains to show that (6) holds.

Assume that $j$ is an $n$-point. Let $I$ be the $\sim_n$-equivalence class of $j$. Then $j$ is the only element of $I$. If $j \in R' \leq n$ then $O'_{n,I}$ is the expansion of $M_j$ by $R$ which holds at every point. Therefore, if $T^n(M_j) = \tau$, then $T^n(O'_{n,I}) = \tau^R$. Hence, (6) holds in this case. The case when $j$ is an $n$-point and $j \not\in R' \leq n$ is similar.

Assume now that $j$ is not an $n$-point. Then the $\sim_n$ equivalence class $I$ of $j$ is an $S$-interval for some $S \subseteq H_n(\Delta)$. Hence, $I$ is order-isomorphic to $\eta$, the predicates $\text{Type}_{n,t}$ (for $t \in S$) partition $I$, and each of these predicates is dense in $I$.

Recall that $R'$ satisfies ($\ast$), thus $R'$ is both dense and co-dense in each $\text{Type}_{n,t} \cap I$ (for $t \in S$). If $j \in \text{Type}_{n,t} \cap I \cap R'$, then $T^n(M_j') = \tau^R$; if $j \in (\text{Type}_{n,t} \cap I) \setminus R'$, then $T^n(M_j') = \tau^{-R}$.

Since $O_{n,I} = \sum_{j \in I} M_j'$, we obtain by Theorem 2.8 that

$$T^n(O'_{n,I}) = \text{shuffle}(\{\tau^R \mid \tau \in S\} \cup \{\tau^{-R} \mid \tau \in S\}).$$

This completes the proof of (6), of Lemma 3.14 and of Proposition 3.13.

Third case follows from Proposition 3.12 and Proposition 3.13.

§4. Further results and open questions. We proved that if the monadic second-order theory of a countable chain $C$ is decidable, then $C$ has a non-trivial expansion with decidable monadic second-order theory.

It would be interesting to obtain a version of our result for first-order logic. However, such a proof requires some new ideas. One obstacle is that there is no first-order formula that expresses that an interval $(x, y)$ is finite. This is expressible in MSO and allowed us to consider three types of intervals.

We also do not know whether the main result of [3] can be extended to first-order logic, namely whether decidability of the first-order theory of a chain which contains an interval of order type $\omega$ or $-\omega$ implies that it has non-trivial expansion with decidable first-order theory.

Another interesting issue is to remove the assumption that the ordering is countable and to prove that every chain $C$ has a non-trivial expansion $C'$ such that the monadic theory of $C'$ is recursive in the monadic theory of $C$. Note that the MSO theory of the real line is undecidable [20].

Acknowledgement. We thank the anonymous referee for useful suggestions.

This research was facilitated by the ESF project AutoMathA. The second author was partially supported by ESF project Games and EPSRC grant.

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